Finite-dimensional Feynman-type integrals

1985 J. Phys. A: Math. Gen. 18 3511

(http://iopscience.iop.org/0305-4470/18/18/014)

J. Phys. A: Math. Gen. **18** (1985) 3511-3519. Printed in Great Britain

Finite-dimensional Feynman-type integrals

H Nencka-Ficek† and Jan Tarski

International Centre for Theoretical Physics, POB 586, 34100 Trieste, Italy

Received 26 February 1985, in final form 20 May 1985

Abstract. Oscillatory Feynman-type integrals over finite-dimensional spaces are considered, together with other related conditionally convergent integrals. A representation theorem related to approximating integrals is presented, and the connection of Feynman integrals to Tauberian theorems is discussed. Some counterexamples are included.

1. Introduction

In the mathematical analysis of Feynman path integrals, one encounters two sources of complications: the integrals are over infinite-dimensional spaces, and their convergence is due to oscillatory rather than decreasing (or damping) factors. In the recent mathematical analysis, the first of these complications was usually approached by regarding the integrals as Gaussian integrals over a real Hilbert space $\mathscr{H}$, which is in effect determined by the scalar product in the oscillatory Gaussian factor. The resulting oscillatory integrals have a more general form than the path integral of Feynman, and they, together with their variants, have been called Feynman-type integrals.

We should like to explore here some aspects of these integrals which relate specifically to their oscillatory convergence. We therefore examine the simpler case, $\dim \mathscr{H} < \infty$. This case indeed constitutes a natural subject for study, even though it may have only a slight physical motivation.

We might point out in this connection that such finite-dimensional integrals do occur in a physical context. In particular, the composition law for Green functions for the Schrödinger equation takes the form (where $0 < s < t$),

$$G(t; y, x) = \int \mathrm{d}^k u \, G(s; u, x) G(t-s; y, u). \tag{1.1}$$

The integral over u is characterised by an oscillatory convergence, and so is of Feynman type. Examples could also be given in which the familiar path integral for G reduces to a finite-dimensional Feynman-type integral. However, we will not be concerned further with such examples, and we regard the present paper primarily as providing some background for future investigations of the infinite-dimensional case.

This paper is based on the definition of Feynman-type integrals which is given in Tarski (1979). For the case $\dim \mathscr{H} = k < \infty$, this definition reduces to the following.

† Permanent address: Institute of Molecular Physics, Polish Academy of Sciences, Smoluchowskiego 17/19, 60-179 Poznań, Poland.

0305-4470/85/183511+09$02.25

Let κ satisfy $\operatorname{Im}\kappa \geqslant 0$, $\kappa \neq 0$, let $\operatorname{Re} b > 0$, let $\alpha \in R^k$ be arbitrary, and let

$$I^{b,\alpha}(f) = [(b - i\kappa)/2\pi]^{k/2} \int d^k u \exp[-\tfrac{1}{2}b\langle u-\alpha, u-\alpha\rangle] \exp[\tfrac{1}{2}i\kappa\langle u, u\rangle] f(u). \tag{1.2}$$

The function $f: R^k \to C^1$ is to be such that the combined integrand is in L_1. Consider now non-tangential limits of $I^{b,\alpha}(f)$ as $b \to 0$ (i.e. along non-tangential curves). If these limits are equal and do not depend on α, then the common limit is the integral in question. This integral has been denoted as follows:

$$\lim_{b\to 0} I^{b,\alpha}(f) = I(f) = \int \mathscr{D}(\xi) \exp[\tfrac{1}{2}i\kappa\langle \xi, \xi\rangle] f(\xi). \tag{1.3}$$

(In typical examples with integrable f, $I^{b,\alpha}(f)$ is bounded near $b = 0$. The non-tangential limits are then necessarily equal (Priwalow 1956 pp 18-9).) We may note that the integral in (1.2) remains meaningful if f is a distribution in the class $\mathscr{S}^{1/2}_{1/2}(R^k)'$ (Gelfand and Shilov 1968). See also the note added at the end of this paper.

For analysis of the infinite-dimensional case one uses finite-dimensional approximations, which have the form of (1.2). In such an analysis it is essential to separate the factor $[(b - i\kappa)/2\pi]^{k/2}$ as above, and it is convenient to keep $\exp[\frac{1}{2}i\kappa\langle u, u\rangle]$ separate. However, if $\dim \mathscr{H} < \infty$, then these factors can be absorbed into f. We set, accordingly,

$$\bar{I}(b, \alpha; f) = \int d^k u \exp[-\tfrac{1}{2}b\langle u-\alpha, u-\alpha\rangle] f(u) \tag{1.4}$$

and we will denote the limit as $b \to 0$ under the above conditions by $\bar{I}$:

$$\lim_{b\to 0} \bar{I}(b, \alpha; f) = \bar{I}(f). \tag{1.5}$$

In this paper we examine (1.2)-(1.5) from the following points of view: (i) the inter-relation between $\bar{I}(b, \alpha; f)$ and $f(u)$ (§ 2), (ii) the connection with Tauberian theorems (§ 3), and (iii) the differences between $\bar{I}$ and the Lebesgue integral, as illustrated by counterexamples (§ 4).

We may note that finite-dimensional examples of Feynman-type integrals were considered on various occasions, especially in Buchholz and Tarski (1976), Tarski (1981) and Elworthy and Truman (1984). The last two papers exhibit the occurrence of the Morse (or Maslov) index in such finite-dimensional integrals.

The definition of Feynman-type integrals which was adopted by Buchholz and Tarski (1976) is a specialisation and a slight modification of that of Itô (1966). We will see in § 4 that these definitions are not equivalent to that specified in (1.2)-(1.5) (or to the infinite-dimensional extension).

2. A representation theorem

In this section we examine $\bar{I}(b, \alpha; f)$ in its dependence on b and α when f is fixed. We consider also the problem of determining f when a suitable $\bar{I}(b, \alpha; f)$ is given, i.e. the problem of representing f in terms of a suitable function $\psi(b, \alpha)$. We first note three properties of $\bar{I}$:

(i) $\bar{I}(b, \alpha; f)$ is analytic in the region $\operatorname{Re} b > 0$, $\alpha \in C^k$. Let us denote $\bar{I}(b, \mathrm{i}\alpha; f)$ for a fixed f by $\psi(b, \alpha)$.

(ii) The function $\psi(b, \alpha)$ satisfies (in the region of analyticity)

$$[2b^2(\partial/\partial b) + kb]\psi = \nabla_\alpha^2 \psi. \tag{2.1}$$

(iii) For $b > 0$ and $\alpha \in R^k$, ψ is of the form $\exp[\frac{1}{2}b\langle\alpha, \alpha\rangle]\psi_1$ where ψ_1 and its derivatives $\partial\psi_1/\partial b$, $\partial\psi_1/\partial\alpha^j$, $\partial^2\psi_1/\partial(\alpha^j)^2$ are Fourier transforms of L_1 functions in their dependence on α.

Proposition 1. If $f(u)\exp[-\frac{1}{2}b\langle u, u\rangle] \in L_1$ for $\forall b > 0$, then (i)-(iii) hold. Conversely, if $\varphi(b, \alpha)$ is defined for $b > 0$ and $\alpha \in R^k$, and satisfies (ii) (for b, α real) and (iii), then there exists a function f, uniquely determined almost everywhere, such that $f(u)\exp[-\frac{1}{2}b\langle u, u\rangle] \in L_1$ for $\forall b > 0$ and $\psi(b, \alpha) = \bar{I}(b, \mathrm{i}\alpha; f)$.

Proof. If $f\exp[-\frac{1}{2}b\langle u, u\rangle] \in L_1$, then (i)-(iii) follow from the bounded convergence theorem, where for (ii) we differentiate under the integral sign.

Conversely, let $\psi(b, \alpha)$ be given, such as specified. We write (with $f_0 \in L_1$, as a function of u)

$$\psi(b, \alpha) = \exp[\tfrac{1}{2}b\langle\alpha, \alpha\rangle] \int \mathrm{d}^k u \exp[\mathrm{i}b\langle u, \alpha\rangle] f_0(b, u). \tag{2.2}$$

Since the relevant derivatives of ψ are assumed to be Fourier transforms of L_1 functions, we may differentiate ψ under the integral sign, and (2.1) yields

$$\exp[\tfrac{1}{2}b\langle\alpha, \alpha\rangle] \int \mathrm{d}^k u \exp[\mathrm{i}b\langle u, \alpha\rangle][2b^2(\partial f_0/\partial b) + b^2\langle u, u\rangle f_0] = 0. \tag{2.3}$$

The resulting equation $[\ldots] = 0$ implies that f_0 has the form $f(u)\exp[-\frac{1}{2}b\langle u, u\rangle]$. The function $f(u)$ is now determined by (2.2).

We remark that an alternative method of obtaining f in terms of ψ depends on recognising $(b/2\pi)^{k/2}\exp[-\frac{1}{2}b\langle u-\alpha, u-\alpha\rangle]$ as the kernel of the heat equation with $b = t^{-1}$. Thus, for suitable f, $\lim_{t\searrow 0}(2\pi t)^{-k/2}\psi(t^{-1}, \alpha) = f(\alpha)$.

We remark also that the above proposition could be helpful in introducing a topology for Feynman-integrable functions. However, it does not seem to elucidate the problem of the limit as $b \to 0$ in a more direct way.

We turn to the case where f is a distribution. In this case the property (iii) does not hold as stated, but suitable generalisations would allow the previous discussion to go through. For example, let us replace (iii) by

(iii′) For $b > 0$ and $\alpha \in R^k$, ψ is of the form $\exp[\frac{1}{2}b\langle\alpha, \alpha\rangle]\psi_1$, where ψ_1 and its derivatives $\partial\psi_1/\partial b$, $\partial\psi_1/\partial\alpha^j$, $\partial^2\psi_1/\partial(\alpha^j)^2$ are polynomially bounded, as functions of α.

We can now adapt proposition 1 and its proof to the new situation, i.e. we specify that $f(u)\exp[-\frac{1}{2}b\langle u, u\rangle] \in \mathscr{S}'$ rather than in L_1, we replace (iii) by (iii′), and we use distribution-theoretic analysis (as for example in Reed and Simon 1975, p 15 ff) rather than the bounded convergence theorem. We obtain

Corollary 2. Proposition 1 remains valid if L_1 is replaced by $\mathscr{S}'$, the property (iii) by (iii′), and 'a function $f\ldots$ almost everywhere' by 'a unique distribution f'.

Let us return to equation (2.1). If we set there $\psi = \exp[\frac{1}{2}b\langle\alpha, \alpha\rangle]\psi_1$, then we obtain the following equation for ψ_1:

$$2b^2(\partial/\partial b)\psi_1 = (2b\langle\alpha, \nabla_\alpha\rangle + \nabla_\alpha^2)\psi_1. \tag{2.4}$$

Here the dimension k does not enter explicitly. This equation may therefore also remain applicable in infinite-dimensional situations.

We conclude this section with a simple example which illustrates the foregoing notions. Let $k = 1$ and $\alpha = \alpha' + \alpha''$, and we construct the following expansion, which is valid for Re $b > 0$:

$$\psi_1(b, \alpha' + \alpha'') = \sum c_n(b, \alpha')(\mathrm{i}b\alpha'')^n/n!. \tag{2.5}$$

We observe that

$$c_n(b, \alpha') = \int_{-\infty}^{\infty} \mathrm{d}u \exp(-\tfrac{1}{2}bu^2)\exp(\mathrm{i}b\alpha' u)f(u)u^n \tag{2.6}$$

which implies

$$\partial_b c_0 = -\tfrac{1}{2}c_2 + \mathrm{i}\alpha' c_1 \tag{2.7}$$

and in fact this equation is a special case of (2.4).

If $\bar{I}(f(u)u^n)$ exists for $n = 0, 1, 2$, then the limit as $b \to 0$ in (2.7) is possible for (at least) $\mathrm{i}\alpha' = \beta \in R^1$, and we obtain

$$\partial_b \bar{I}(b, \beta; f)_{b=0} = \bar{I}(-\tfrac{1}{2}(u-\beta)^2 f(u)). \tag{2.8}$$

Here the derivative is in the sense of a non-tangential limit.

3. Feynman-type integrals and Tauberian theorems

Tauberian theorems relate to limits like the following:

$$\lim_{x\to\infty} \int_0^\infty \mathrm{d}t \exp(-t/x)f(t) \tag{3.1}$$

and in many formulae there is an additional factor x^{-1} in front of the integral. Similarity to the Feynman integrals is striking, and the latter integrals can be trivially reduced to limits as in (3.1). However, the passage in the opposite direction requires some additional discussion.

The analysis of Dunford and Schwartz (1963) shows that if $f(t)$ and $tf(t)$ are bounded, and the limit in (3.1) exists, then

$$\lim_{x\to\infty} \int_0^\infty \mathrm{d}t \exp(-t/x)f(t) = \lim_{x\to\infty} \int_0^x \mathrm{d}t f(t). \tag{3.2}$$

(We note that we have to eliminate cases where f is a distribution and not a function.) Some simple remarks were made in Buchholz and Tarski (1976) and Tarski (1976) concerning (3.2) and its connection with Feynman integrals. We summarise the situation as follows.

Lemma 3. (a) Both limits in (3.2) exist, and the equation is valid, if one of these conditions is fulfilled: (a_1) $f(t)$ and $tf(t)$ are bounded, and the LHS exists; or (a_2)

$f(t)\exp(-t/x)$ is in L_1 for $\forall x>0$, f has a finite number of changes of sign in any finite interval, and the RHS exists. (b) Let $g: R^k \to C^1$ be such that $\bar{I}(g)$ exists and such that $r^k g(u)$ and $r^{k-2} g(u)$ are bounded, where $r^2 = \Sigma_j (u^j)^2$. Then

$$\bar{I}(g) = \lim_{R\to\infty} \int_0^R \mathrm{d}r\, r^{k-1} \int_{S^{k-1}} \mathrm{d}^{k-1}\Omega g(r,\Omega) \tag{3.3}$$

where Ω refers to the angular variables.

The proof of part (a_2) is elementary. We will refer to this part in the subsequent discussion.

We turn to the problem of establishing a criterion for Feynman integrability in terms of limits like (3.1). For simplicity we will confine ourselves here to integration over R^1. We recall that for Feynman integrability, we have to allow the parameter $b = x^{-1}$ to be complex, and to incorporate the shift vector, or parameter, α.

The parameter α gives the approximating integrals the form of a convolution in R^1, while the parameter b, when real, relates to a convolution in R^1_+. It is therefore tempting to analyse the Feynman integral in terms of convolution algebras. We will not attempt such a study here, but rather we will prove proposition 4 below with the help of an elementary estimate. Our first step is to extend limits like (3.1) to complex parameters.

Lemma 4. Let f, $g \in L_\infty(R^1)$, let $uf(u)$ be bounded, and let $a \in R^1$. Then

$$\lim_{x\to\infty} x^{-1} \int_0^\infty \mathrm{d}t \exp(-t/x) g(t) = \lim_{x\to\infty} (1+\mathrm{i}a) x^{-1} \int_0^\infty \mathrm{d}t \exp[-(1+\mathrm{i}a)t/x] g(t) \tag{3.4}$$

$$\lim_{x\to\infty} \int_0^\infty \mathrm{d}s \exp(-s/x) f(s) = \lim_{x\to\infty} \int_0^\infty \mathrm{d}s \exp[-(1+\mathrm{i}a)s/x] f(s). \tag{3.5}$$

In each equation the existence of one limit implies that of the other and the equality.

Proof. For (3.4), we follow Dunford and Schwartz (1963) (cf also the discussion in Wiener (1964)). Contour integration yields, for $a \in R^1$,

$$\int_0^\infty (\mathrm{d}t/t) \exp(-t) t^{\mathrm{i}x} = \Gamma(\mathrm{i}x) = (1+\mathrm{i}a)^{\mathrm{i}x} \int_0^\infty (\mathrm{d}t/t) \exp[-(1+\mathrm{i}a)t] t^{\mathrm{i}x}. \tag{3.6a,b}$$

But $\Gamma(\mathrm{i}x) \neq 0$. Therefore, if the LHS of (3.4) exists, (3.6a) yields the desired conclusion, while if the RHS exists, (3.6b) can be similarly applied. For (3.5), we first observe that the assumed existence of one limit and the assumed bounds imply that $F(t) = \int_0^t \mathrm{d}s\, f(s)$ is bounded, cf Dunford and Schwartz (1957, 1963). By inserting $g(t) = F(t)$ in (3.4), interchanging integrations, and doing the t integrals, we obtain (3.5).

Proposition 5. Let f, $uf(u)$ be bounded, let f have a finite number of changes of sign in any finite interval, and assume that

$$\lim_{x\to\infty} \int_{-\infty}^\infty \mathrm{d}u \exp(-u^2/2x) f(u) \tag{3.7}$$

exists. Then $\bar{I}(f)$ exists.

Before proving this proposition we make two comments. First, example 5 of § 4 is one where the hypothesis is fulfilled except for boundedness of $uf(u)$, and where $\bar{I}(f)$ does not exist. Second, the hypothesis regarding the changes of sign of f can be dropped, and the application of part (a_2) of lemma 3 can be bypassed, if we assume in addition that the following limits exist for $\forall\alpha \in R^1$:

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} du \exp[-(u-\alpha)^2/2x]f(u). \tag{3.8}$$

Proof. Let us first show that the limits in (3.8) exist and are equal. For definiteness we will assume $x > \alpha \geqslant 0$. Then

$$\begin{aligned}\lim_{x\to\infty} \int_{-\infty}^{\infty} du \exp(-u^2/2x)f(u) &= \lim \int_{-x}^{x} du f(u)\\ &= \lim\left(\int_{-x-\alpha}^{x-\alpha} du f(u) + \mathcal{O}(x^{-1})\right) = \lim \int_{-x}^{x} du f(u+\alpha)\\ &= \lim \int_{-\infty}^{\infty} du \exp(-u^2/2x) f(u+\alpha) = \lim \int_{-\infty}^{\infty} du \exp[-(u-\alpha)^2/2x]f(u).\end{aligned} \tag{3.9}$$

Next, lemma 4 and a compactness argument show that for a given α, one has a non-tangential limit as $b \to 0$, necessarily equal to the $\lim(x\to\infty)$ above.

4. Counterexamples

It should be useful to have some examples where the integral I, or $\bar{I}$, behaves in a way which differs from what may be expected by a naive comparison with L_1 convergence. Of the following examples, the first three are new, and the last two are included for completeness.

4.1. Converse to Fubini's theorem

If $f_j(u_j)$, $j = 1, \ldots, n$, are integrable for $\bar{I}$, then so is $\Pi_j f_j(u_j)$, but the converse does not hold in general. Let $f = 1$. Then $\bar{I}(b, \alpha; f) = (2\pi/b)^{1/2}$, independently of α. Consider $P_2(u) = \frac{1}{2}(3u^2 - 1)$, the second Legendre polynomial. Let $g(u) = P_2(u)$ for $-1 < u < 1$ and 0 otherwise. Expansion of the exponential then yields

$$\bar{I}(b, \alpha; g) = -\tfrac{2}{15}b + \mathcal{O}(b^2) \tag{4.1}$$

and the contribution proportional to $b\alpha$ vanishes by symmetry. Therefore, for $F(u_1, u_2, u_3) = f(u_1)f(u_2)g(u_3)$, we have $\bar{I}(F) = -4\pi/15$, while the individual factors are not all integrable for $\bar{I}$.

4.2. Invariance upon linear transformations

The integral $\bar{I}$, or I, is clearly invariant under an orthogonal transformation of R^k, but this conclusion does not extend to a general linear transformation. Indeed, taking f,

g and F as above, let

$$G(w_1, \ldots, w_4, u, v) = f(w_1) \ldots f(w_4) g(u) g(v) \tag{4.2}$$

and consider the transformations $u \to u$, $v \to v \pm u$, with the w_j left unchanged. Note that the Jacobian equals unity. Let G_0 be the transformed function:

$$G_0(w_1, \ldots, w_4, u, v) = f(w_1) \ldots f(w_4) g(u) g(u+v). \tag{4.3}$$

It is trivial that $\bar{I}(G) = [\bar{I}(F)]^2$, but the integrals of G and of G_0 differ:

$$\bar{I}(G_0) = 4\bar{I}(G). \tag{4.4}$$

To obtain this evaluation, we express the integral of G_0 in terms of the variables u and $y = u + v$. One can check that when the cut-off factor $\exp(-\frac{1}{2}b \ldots)$ is expanded, only the term proportional to u^2y^2 will contribute to $\bar{I}(G_0)$, and only the term proportional to u^2v^2 will contribute to $\bar{I}(G)$. Thus, there will be equal contributions from $2u^2(y-u)^2$ and from $2u^2v^2$, but there will be an extra contribution to $\bar{I}(G_0)$ from $v^4 = (y-u)^4 = 6y^2u^2 + \ldots$.

Equation (4.4) expresses the non-invariance of $\bar{I}$ upon linear transformations. This fact has the following consequence.

Proposition 6. The definition of $I(f)$, given in Tarski (1979), and the definition given in Itô (1966) of the corresponding integral, are inequivalent.

Proof. The definition of Itô (1966) implies invariance under non-singular linear transformations (subject to trace conditions). Consequently, G and G_0, when modified by the factor $\exp[-\frac{1}{2}i\kappa\langle\xi, \xi\rangle]$ to cancel the original $\exp[\frac{1}{2}i\kappa\langle\xi, \xi\rangle]$, are not integrable in the sense of Itô (1966). However, they are integrable in the sense of Tarski (1979).

In fact, in the finite-dimensional case, we can make the following stronger statement: the definition of Buchholz and Tarski (1976) (a variant of that in Itô (1966)) is more restrictive than the definition of equations (1.2)-(1.5).

4.3. *Interchangeability of limits*

In view of the subtle limiting procedure for $\bar{I}$, we expect that counterexamples to interchanging limits and integration should be easy to construct. We give here a simple example. Take

$$f_n(u) = c_n \exp(\tfrac{1}{2}is_n u^2). \tag{4.5}$$

If $c_n \to 0$ with $s_n =$ constant, then $\bar{I}(f_n) \to 0$, while if $s_n \to 0$ with $c_n =$ constant, then $\bar{I}(f_n)$ diverges. It is clear that by letting $c_n \to 0$ and $s_n \to 0$ in a suitable way, any value for the limit of $\bar{I}(f_n)$ can be obtained. If this value is not equal to 0, then

$$\lim_{n\to\infty} \bar{I}(f_n) \neq \bar{I}(\lim f_n) = \bar{I}(0) = 0. \tag{4.6}$$

Note that $|f_n(u)| \leq \max |c_n|$, which is an integrable function for I (of (1.3)).

4.4. *Boundedness by an integrable function and integrability*

The following example was given in Tarski (1975), with reference to the integrals $I^{b,\alpha}(f)$ and $I(f)$ on R^1. We start with $F(u) = u^2$, which is integrable for I. Let $\kappa > 0$, and for the real part of I replace $\exp(\frac{1}{2}\mathrm{i}\kappa u^2)$ by $\cos(\frac{1}{2}\kappa u^2)$. Let $u_n > 0$ be defined by $\frac{1}{2}\kappa u_n^2 = 2n + \frac{3}{2}$, for n integral and $n \geq 0$. Let $f(u) = 0$ for $u < u_0$, and let $f(u) = u_n^2$ for $u_n \leq u < u_{n+1}$. Then $|f(u)| \leq u^2$. However, one cycle of $\cos(\frac{1}{2}\kappa u^2)$ yields a contribution whose positive part is larger than the negative, and the excess is sufficient to cause $I(f)$ to diverge. (A formal proof depends on verifying that the RHS of (3.2) diverges, and on adapting lemma 3, part (a_2), to infinite values of integrals.)

4.5. *Invariance under translations*

This condition is equivalent to having the limit of $\bar{I}(b, \alpha; f)$ independent of α, as $b \to 0$. However, an α-dependent limit is obtained for the function $f(u) = \varepsilon(u)$ on R^1, where $\varepsilon(u) = \pm 1$ for $u \gtrless 0$. This example is fully discussed in Tarski (1980).

Acknowledgments

The authors would like to express their appreciation to Professor Abdus Salam, UNESCO and IAEA for hospitality at the International Centre for Theoretical Physics. One of us (HN-F) is grateful to the administration of Research Project No MRI 9 for partial support.

Note added. The extension of α to the complex region, as in § 2, suggests that the following (non-tangential) limits might also be of interest:

$$\lim_{b \to 0} \int \mathrm{d}^k u \exp[-\tfrac{1}{2} b \langle u - \alpha - \mathrm{i}\beta, u - \alpha - \mathrm{i}\beta \rangle] f(u) \tag{N1}$$

for α, $\beta \in R^k$. (Here the scalar product is symmetric, and not Hermitian.) In this note we will elaborate on such integrals and limits.

We recall that in the real case ($\beta = 0$), independence of α is equivalent to translational invariance, and on the physical side it allows a derivation of the Schwinger action principle (Tarski 1979, 1980). We will see that independence of α and β in the limit similarly is equivalent to invariance with respect to complex translations. However, it is not clear if such additional invariance has any new physical significance.

As we indicated earlier, we may regard f in (N1) as an element of $\mathcal{S}_{1/2}^{1/2}(R^k)'$. Since functions of $\mathcal{S}_{1/2}^{1/2}(R^k)$ extend to entire functions on C^k (Gelfand and Shilov 1968), such a conclusion applies also to distributions in $\mathcal{S}_{1/2}^{1/2}(R^k)'$. One way of expressing the continuation depends on Fourier transformation (cf Gelfand and Shilov 1968):

$$f(u + \mathrm{i}v) = (2\pi)^{-k/2} \int \mathrm{d}^k p \exp[-\mathrm{i}p(u + \mathrm{i}v)] \tilde{f}(p). \tag{N2}$$

(Note: $f = \mathcal{S}_{1/2}^{1/2\prime}$ implies $\tilde{f} \in \mathcal{S}_{1/2}^{1/2\prime}$, implies $\exp(pv)\tilde{f} \in \mathcal{S}_{1/2}^{1/2\prime}$, implies $f(u + \mathrm{i}v) \in \mathcal{S}_{1/2}^{1/2\prime}$.) It then follows directly that

$$\int \mathrm{d}^k u \exp[-\tfrac{1}{2} b \langle u - \alpha - \mathrm{i}\beta, u - \alpha - \mathrm{i}\beta \rangle] f(u) = \int \mathrm{d}^k u \exp[-\tfrac{1}{2} b \langle u, u \rangle] f(u + \alpha + \mathrm{i}\beta). \tag{N3}$$

These considerations lead to the following:

Proposition 7. Assume that $\bar{I}(f)$ exists. Then the non-tangential limit $b \to 0$ in (N1) is independent of α and β if and only if $\bar{I}(f(u + \mathrm{i}\beta))$ exists and is equal to $\bar{I}(f)$ for $\forall \beta \in R^k$. (Equality with $\bar{I}(f(u + \alpha + \mathrm{i}\beta))$ then follows.) The independence and the invariance are fulfilled in particular for: (i) $f \in L_1(R^k)$, (ii) functions f which are of the form $\exp[\frac{1}{2}\mathrm{i}\kappa\langle u, u\rangle]$ times the Fourier transform of a distribution (cf below), (iii) functions f which are of the form $\exp[\frac{1}{2}\mathrm{i}\kappa\langle u, u\rangle]$ times the restriction to R^k of an entire function of order less than 2. In case (iii), $f(u + \mathrm{i}v)$ given by (N2) agrees with the result obtained by the usual (function-theoretic) analytic continuation.

Proof. The equivalence of indepndence and invariance follows directly from (N3), and case (i) is trivial. Next, the last assertion (about $f(u+iv)$) follows by first interpreting (N3) in terms of contour integration (this shows that both methods of continuation make (N3) valid), and second, by observing that (N3) remains valid if, in place of $\exp[-\frac{1}{2}b\langle u, u\rangle]$, we use an arbitrary function in $\mathscr{S}^{1/2}_{1/2}(R^k)$ (cf Gelfand and Shilov 1968, especially §§ 2, 7 and 9). Case (iii) then follows by contour integration (cf Buchholz and Tarski 1976).

There remains case (ii). The relevant functions are of the form (Berg and Tarski 1981)

$$f(u)=\exp[\tfrac{1}{2}\mathrm{i}\kappa\langle u, u\rangle](-1)^n\int \mathrm{d}\mu(w)(\partial/\partial w^{j_1})\dots(\partial/\partial w^{j_n})\exp[\mathrm{i}\langle w, u\rangle] \qquad (\mathrm{N}4a)$$

where

$$\int \mathrm{d}|\mu|(w)(1+|w^{j_1}|)\dots(1+|w^{j_n}|)<\infty. \qquad (\mathrm{N}4b)$$

We now recall that the proof for the case $\beta=0$ proceeds by first integrating $\exp[\mathrm{i}\langle w, u\rangle]$ and the Gaussian weights with respect to u. The operators $\partial/\partial w^{j_i}$ can be interchanged with u integration for $\mathrm{Re}\, b>0$. We next obtain a bound on the expression $\exp[\frac{1}{2}(\mathrm{i}\kappa-b)^{-1}\langle w, w-2\mathrm{i}b\alpha\rangle]$. This bound allows the interchange of $\lim(b\to 0)$ with the w integration, in view of (N4b) and the bounded convergence theorem.

In the present case the foregoing expression becomes

$$\exp[\tfrac{1}{2}(\mathrm{i}\kappa-b)^{-1}\langle w, w-2\mathrm{i}b\alpha+2b\beta\rangle]. \qquad (\mathrm{N}5)$$

Here κ and b are allowed to be complex. The needed bound may depend on κ, α and β, but not on $w\in R^k$ nor on b. However, for a non-tangential limit we suppose

$$|\mathrm{Im}\, b|\leq c\,\mathrm{Re}\, b\leq B. \qquad (\mathrm{N}6)$$

(We allow here arbitrary $c>0$, and assume some $B>0$.) We recall that $\mathrm{Im}\,\kappa\geq 0$ and easily verify that (with constant >0)

$$|\exp[\tfrac{1}{2}(\mathrm{i}\kappa-b)^{-1}\langle w, w\rangle]|\leq\exp[-(\mathrm{Re}\, b)\|w\|^2(\mathrm{constant})] \qquad (\mathrm{N}7a)$$

$$|\exp[\tfrac{1}{2}(\mathrm{i}\kappa-b)^{-1}\langle w, -2\mathrm{i}b\alpha+2b\beta\rangle]|\leq\exp[(\mathrm{Re}\, b)\|w\|(\mathrm{constant})]. \qquad (\mathrm{N}7b)$$

It follows that the product (N5) has a bound independent of b (but dependent on B and c). This enables the argument of Berg and Tarski (1981), as outlined above, to go through.

We remark that if $\beta=0$, then it is not necessary to make the restriction $|\mathrm{Im}\, b|\leq c\,\mathrm{Re}\, b$, and tangential limits also give the desired result.

References

Berg H P and Tarski J 1981 *J. Phys. A: Math. Gen.* **14** 2207

Buchholz D and Tarski J 1976 *Ann. Inst. H Poincaré* A **24** 323 and *Preprint* DESY 74/35

Dunford N and Schwartz J T 1957 *Linear Operators, Part I* (New York: Wiley-Interscience) p 78, exercise II.4.53

—— 1963 *Linear Operators, Part II* (New York: Wiley-Interscience) pp 1003–7

Elworthy D and Truman A 1984 *Ann. Inst. H Poincaré* A **41** 115

Gelfand I M and Shilov G E 1968 *Generalized Functions* vol 2 (New York: Academic) ch IV

Itô K 1966 *Proc. 5th Berkeley Symp. on Mathematical Statistics and Probability* vol II, part I (Berkeley, CA: University of California Press) p 145

Priwalow I I 1956 *Randeigenschaften Analytischer Funktionen* (Berlin: VEB Deutscher Verlag der Wissenschaften) 2nd edn

Reed M and Simon B 1975 *Methods of Modern Mathematical Physics* vol II (New York: Academic)

Tarski J 1975 *Functional Integration and Its Applications* ed A M Arthurs (Oxford: OUP) p 169

—— 1976 *Complex Analysis and Its Applications* vol III (Vienna: IAEA) p 193

—— 1979 *Feynman Path Integrals. Conf. Proc. Marseille, 1978* (*Lecture Notes in Physics* **106**) ed S Albeverio, Ph Combe, R Høegh-Krohn, G Rideau, M Sirugue-Collin, M Sirugue and R Stora (Berlin: Springer) p 254

—— 1980 *Functional Integration: Theory and Applications* ed J-P Antoine and E Tirapegui (New York: Plenum) p 143

—— 1981 *ICTP, Trieste Rep.* SMR/92-29

Wiener N 1964 *Selected Papers of Norbert Wiener* (Cambridge, MA: MIT Press) p 261